

# Canonical superenergy tensors in general relativity: a reappraisal

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Here we present our actual point of view on the canonical superenergy tensors.

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## I. INTRODUCTION

In the framework of general relativity (**GR**), as a consequence of the Einstein Equivalence Principle (**E**EP, the gravitational field *has non-tensorial strengths*  $\Gamma_{kl}^i = \{^i_{kl}\}$  and *admits no energy-momentum tensor*. One can only attribute to this field *gravitational energy-momentum pseudotensors*. The leading object of such a kind is the *canonical gravitational energy-momentum pseudotensor*  $Et_i^k$  proposed already in past by Einstein. This pseudotensor is a part of the *canonical energy-momentum complex*  ${}_{EK_i}^k$  in **GR**.

The canonical complex  ${}_{EK_i}^k$  can be easily obtained by rewriting Einstein equations to the superpotential form

$${}_{EK_i}^k := \sqrt{|g|}(T_i^k + {}_E t_i^k) = {}_F U_i^{[kl]}{}_{,l} \quad (1)$$

where  $T^{ik} = T^{ki}$  is the symmetric energy-momentum tensor for matter,  $g = \det[g_{ik}]$ , and

$$\begin{aligned} Et_i^k &= \frac{c^4}{16\pi G} \left\{ \delta_i^k g^{ms} (\Gamma_{mr}^l \Gamma_{sl}^r - \Gamma_{ms}^r \Gamma_{rl}^l) \right. \\ &\quad + g^{ms}{}_{,i} [\Gamma_{ms}^k - \frac{1}{2} (\Gamma_{tp}^k g^{tp} - \Gamma_{tl}^l g^{kt})] g_{ms} \\ &\quad \left. - \frac{1}{2} (\delta_s^k \Gamma_{ml}^l + \delta_m^k \Gamma_{sl}^l) \right\}; \end{aligned}$$

$${}_F U_i^{[kl]} = \frac{c^4}{16\pi G} g_{ia} (\sqrt{|g|})^{(-1)} [(-g)(g^{ka} g^{lb} - g^{la} g^{kb})]_{,b}. \quad (2)$$

$Et_i^k$  are components of the canonical energy-momentum pseudotensor for gravitational field  $\Gamma_{kl}^i = \{^i_{kl}\}$ , and  ${}_F U_i^{[kl]}$  are von Freud superpotentials.

$${}_{EK_i}^k = \sqrt{|g|}(T_i^k + {}_E t_i^k) \quad (3)$$

are components of the *Einstein canonical energy-momentum complex*, for matter and gravity, in **GR**.

Symbol  ${}_{,l}$  means partial derivative  $\partial_l$ .

In the consequence of (1) the complex  ${}_{EK_i}^k$  satisfies local conservation laws

$${}_{EK_i}^k{}_{,k} \equiv 0. \quad (4)$$

In very special cases one can obtain from these local conservation laws reasonable integral conservation laws.

Despite that one can easily introduce in **GR** the *canonical (and others) superenergy tensor* for gravitational field. This was done in past in a series of our articles (See, e.g., [1] and references therein).

It appeared that the idea of the superenergy tensors is universal: to any physical field having an energy-momentum tensor or pseudotensor one can attribute the corresponding superenergy tensor.

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## II. THE CANONICAL SUPERENERGY TENSORS

Here we give a short reminder of the general, constructive definition of the superenergy tensor  $S_a^b$  applicable to gravitational field and to any matter field. The definition uses *locally Minkowskian structure* of the spacetime in **GR** and, therefore, it fails in a spacetime with torsion, e.g., in Riemann-Cartan spacetime.

In the normal Riemann coordinates **NRC(P)** we define (pointwise)

$$S_{(a)}^{(b)}(P) = S_a^b := (-) \lim_{\Omega \rightarrow P} \frac{\int_{\Omega} [T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P)] d\Omega}{1/2 \int_{\Omega} \sigma(P; y) d\Omega}, \quad (5)$$

where

$$\begin{aligned} T_{(a)}^{(b)}(y) &:= T_i^k(y) e_{(a)}^i(y) e_k^{(b)}(y), \\ T_{(a)}^{(b)}(P) &:= T_i^k(P) e_{(a)}^i(P) e_k^{(b)}(P) = T_a^b(P) \end{aligned}$$

are *physical or tetrad components* of the pseudotensor or tensor field which describes an energy-momentum distribution, and  $\{y^i\}$  are normal coordinates.  $e_{(a)}^i(y), e_k^{(b)}(y)$  mean an orthonormal tetrad  $e_{(a)}^i(P) = \delta_a^i$  and its dual  $e_k^{(a)}(P) = \delta_k^a$  parallelly propagated along geodesics through  $P$  ( $P$  is the origin of the **NRC(P)**).

We have

$$e_{(a)}^i(y) e_i^{(b)}(y) = \delta_a^b. \quad (6)$$

For a sufficiently small 4-dimensional domain  $\Omega$  which surrounds **P** we require

$$\int_{\Omega} y^i d\Omega = 0, \quad \int_{\Omega} y^i y^k d\Omega = \delta^{ik} M, \quad (7)$$

where

$$M = \int_{\Omega} (y^0)^2 d\Omega = \int_{\Omega} (y^1)^2 d\Omega = \int_{\Omega} (y^2)^2 d\Omega = \int_{\Omega} (y^3)^2 d\Omega, \quad (8)$$

is a common value of the moments of inertia of the domain  $\Omega$  with respect to the subspaces  $y^i = 0$ , ( $i = 0, 1, 2, 3$ ).

We can take as  $\Omega$ , e.g., a sufficiently small analytic ball centered at  $P$ :

$$(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2, \quad (9)$$

which for an auxiliary positive-definite metric

$$h^{ik} := 2v^i v^k - g^{ik}, \quad (10)$$

can be written in the form

$$h_{ik} y^i y^k \leq R^2. \quad (11)$$

A fiducial observer **O** is at rest at the beginning **P** of the used Riemann normal coordinates **NRC(P)** and its four-velocity is  $v^i = * \delta_o^i$ .  $= *$  means that an equations is valid only in special coordinates.

$\sigma(P; y)$  denotes the two-point *world function* introduced in past by J.L. Synge [2]

$$\sigma(P; y) = * \frac{1}{2} (y^0^2 - y^1^2 - y^2^2 - y^3^2). \quad (12)$$

The world function  $\sigma(P; y)$  can be defined covariantly by the *eikonal-like equation* [2]

$$g^{ik} \sigma_{,i} \sigma_{,k} = 2\sigma, \quad \sigma_{,i} := \partial_i \sigma, \quad (13)$$

together with

$$\sigma(P; P) = 0, \quad \partial_i \sigma(P; P) = 0. \quad (14)$$

The ball  $\Omega$  can also be given by the inequality

$$h^{ik}\sigma_{,i}\sigma_{,k} \leq R^2. \quad (15)$$

Tetrad components and normal components are equal at  $\mathbf{P}$ , so, we will write the components of any quantity attached to  $\mathbf{P}$  without tetrad brackets, e.g., we will write  $S_a^b(P)$  instead of  $S_{(a)}^{(b)}(P)$  and so on.

If  $T_i^k(y)$  are the components of an energy-momentum tensor of matter, then we get from (5)

$${}_m S_a^b(P; v^l) = (2\hat{v}^l \hat{v}^m - \hat{g}^{lm}) \nabla_l \nabla_m \hat{T}_a^b = \hat{h}^{lm} \nabla_l \nabla_m \hat{T}_a^b. \quad (16)$$

Hat over a quantity denotes its value at  $\mathbf{P}$ , and  $\nabla$  means covariant derivative.

Tensor  ${}_m S_a^b(P; v^l)$  is the *canonical superenergy tensor for matter*.

For the gravitational field, substitution of the canonical Einstein energy-momentum pseudotensor as  $T_i^k$  in (5) gives

$${}_g S_a^b(P; v^l) = \hat{h}^{lm} \hat{W}_a^b{}_{lm}, \quad (17)$$

where

$$\begin{aligned} W_a^b{}_{lm} = & \frac{2\alpha}{9} [B_{alm}^b + P_{alm}^b \\ & - \frac{1}{2} \delta_a^b R^{ijk}{}_m (R_{ijkl} + R_{ikjl}) + 2\delta_a^b \beta^2 E_{(l|g} E^g{}_{|m)} \\ & - 3\beta^2 E_{a(l|} E^b{}_{|m)} + 2\beta R^b{}_{(a|g|l)} E^g{}_m]. \end{aligned}$$

Here  $\alpha = \frac{c^4}{16\pi G} = \frac{1}{2\beta}$ , and

$$E_i^k := T_i^k - \frac{1}{2} \delta_i^k T \quad (18)$$

is the modified energy-momentum tensor of matter [6].

On the other hand

$$B_{alm}^b := 2R^{bik}{}_{(l|} R_{aik|m)} - \frac{1}{2} \delta_a^b R^{ijk}{}_l R_{ijkm} \quad (19)$$

are the components of the *Bel-Robinson tensor (BRT)*, while

$$P_{alm}^b := 2R^{bik}{}_{(l|} R_{aki|m)} - \frac{1}{2} \delta_a^b R^{jik}{}_l R_{jkim} \quad (20)$$

is the Bel-Robinson tensor with “transposed” indices  $(ik)$ .

Tensor  ${}_g S_a^b(P; v^l)$  is the *canonical superenergy tensor* for gravitational field  $\{\hat{C}_{kl}^i\}$ .

In vacuum  ${}_g S_a^b(P; v^l)$  takes the simpler form

$${}_g S_a^b(P; v^l) = \frac{8\alpha}{9} \hat{h}^{lm} (\hat{C}^{bik}{}_{(l|} \hat{C}_{aik|m)} - \frac{1}{2} \delta_a^b \hat{C}^{i(kp)}{}_{(l|} \hat{C}_{ikp|m)}). \quad (21)$$

Here  $C_{blm}^a$  denote components of the *Weyl tensor*.

Some remarks are in order:

1. in vacuum the quadratic form  ${}_g S_a^b v^a v_b$ , where  $v^a v_a = 1$  is *positive-definite*. This form gives the gravitational *superenergy density*  $\epsilon_g$  for a fiducial observer  $\mathbf{O}$ .
2. In general, the canonical superenergy tensors are uniquely determined only along the world line of the observer  $\mathbf{O}$ . But in special cases, e.g., in Schwarzschild spacetime or in Friedman universes, when there exists a physically and geometrically distinguished four-velocity field  $v^i(x)$ , one can introduce in an unique way the unambiguous fields  ${}_g S_i^k(x; v^l)$  and  ${}_m S_i^k(x; v^l)$ .
3. It can be shown that the superenergy densities  $\epsilon_g, \epsilon_m$ , which have dimension  $\frac{\text{Joul}}{\text{metres}}$ , exactly correspond to the Appel's *energy of acceleration*  $\frac{1}{2} \vec{a} \vec{a}$ .

The Appel's energy of acceleration plays fundamental role in Appel's approach to classical mechanics [3].

4. We have proposed in our previous papers to use the tensor  ${}_g S_i^k(P; v^l)$  as a substitute of the non-existing gravitational energy-momentum tensor.
5. In past we have used the canonical superenergy tensors  ${}_g S_i^k$  and  ${}_m S_i^k$  to local (and also to global) analysis of some well-known solutions to the Einstein equations like Schwarzschild, Kerr, Friedman, Gödel, Kasner, Bianchi I, de Sitter and anti-de Sitter solutions. The obtained results were very interesting (See, [1]), e.g., in Gödel universes the sign of the superenergy density  $\epsilon_s := \epsilon_g + \epsilon_m$  depends on causality ( $\epsilon_s < 0$ ) and non-causality ( $\epsilon_s > 0$ ), and, in Schwarzschild spacetime the integral exterior superenergy  $S$  is connected with Hawking temperature  $T$  of the Schwarzschild black hole:  $S = \frac{8\pi k c^3}{9\hbar G} T$ . We have also studied the transformational rules for the canonical superenergy tensors under conformal rescaling of the metric  $g_{ik}(x)$  [1, 4].
6. The idea of the superenergy tensors can be extended on angular momentum (See, [1]). The obtained angular supermomentum tensors *do not depend* on a radius vector and, in gravitational case, they depend only on “spinorial part” of the suitable gravitational angular momentum pseudotensor.
7. As a result of an averaging the tensors  ${}_g S_a^b(P; v^l)$  and  ${}_m S_a^b(P; v^l)$ , in general, do not satisfy any local conservation laws. Only in a symmetric spacetime or in a spacetime which has constant curvature one can get

$$[{}_g S_a^b(P; v^l)]_{,b} = 0. \quad (22)$$

8. There exists exchange of the canonical superenergy between gravity and matter in the following sense. Let us consider the consequence of the equations (4)

$$(\Delta_E^{(4)} K_i^k)_{,k} = [(\Delta^{(4)}(\sqrt{|g|} t_i^k) + \Delta^{(4)}(\sqrt{|g|} T_i^k))]_{,k} = 0, \quad (23)$$

where  $\Delta^{(4)} := (\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2$ .

The exchanged quantities (with total balance equal to zero)

$$\Delta^{(4)}(\sqrt{|g|} t_i^k), \quad \Delta^{(4)}(\sqrt{|g|} T_i^k) \quad (24)$$

have dimensions of the canonical superenergy and, when taken at the beginning  $\mathbf{P}$  of the  $\mathbf{NRC}(\mathbf{P})$  and written covariantly, then they coincide with the canonical superenergy tensors  ${}_g S_i^k(P; v^l)$ ,  ${}_m S_i^k(P; v^l)$  respectively.

Changing the constructive definition (5) to the form

$$< T_a^b(P) > := \lim_{\varepsilon \rightarrow 0} \frac{\int_{\Omega} [T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P)] d\Omega}{\varepsilon^2/2 \int_{\Omega} d\Omega}, \quad (25)$$

where  $\varepsilon := \frac{R}{L} > 0$  (equivalently  $R = \varepsilon L$ ) is a real parameter and  $L$  is a dimensional constant:  $[L] = m$ , one obtains the averaged relative energy-momentum tensors. Namely, from (25) one obtains:  
for matter

$$< {}_m T_a^b(P; v^l) > = {}_m S_a^b(P; v^l) \frac{L^2}{6}, \quad (26)$$

and for gravity

$$< {}_g t_a^b(P; v^l) > = {}_g S_a^b(P; v^l) \frac{L^2}{6}. \quad (27)$$

The components of the averaged relative energy-momentum tensors have correct dimensions, i.e., they have the same dimensions as the components of an energy-momentum tensor but they depend on a dimensional parameter  $L$ . So, introducing of the tensors of such a kind leads us to serious problem, how to choose the dimensional parameter  $L$ ?

It is seen from (26) and (27) that the averaged tensors  $< {}_m T_a^b(P; v^l) >$  and  $< {}_g t_a^b(P; v^l) >$ , for matter and gravitation, can be interpreted as *fluxes* of the appropriate canonical superenergy.

In the paper [1] we have proposed an universal choose of the parameter  $L$ . Namely, we have proposed  $L = 100 L_P \approx 10^{-33} m$ . Here  $L_P := \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35} m$  is the *Planck length*.

Such choice of  $L$  gives the averaged relative energy-momentum tensors which components are negligible in comparison with components of an energy-momentum tensor for matter. In consequence, with such choice of the parameter  $L$ , these tensors play no role in evolution of the material objects and in evolution of the Universe.

On the other hand the choices:

1. For a closed system of the mass  $M$

$$L = \frac{2GM}{c^2}; \quad (28)$$

2. For a gravitational wave of the length  $\lambda$

$$L = \lambda; \quad (29)$$

3. In cosmology

$$L = \frac{2GM_U}{c^2} = \frac{c}{H_o} = ct_o. \quad (30)$$

lead us th the averaged relative energy densities of the same order as ordinary energy density of matter  $\epsilon = T_{ik}v^i v^k$  for an observer which four-velocity is  $v^i$ . Here  $M_U$ ,  $H_o$ ,  $t_o$  mean mass of the observed part of the Universe, actual value of the Hubble constant and the approximated age of the Universe respectively.

So, in this case we have problem how to utilize the averaged relative energy-momentum tensor for matter  $<_m T_a^b(P; v^l) >$  because we already have the tensor  $_m T_a^b(P)$ .

Of course, there exist other possibilities of choosing of the length parameter  $L$ .

In consequence, now we think that the introducing of the one-parameter family of the averaged relative energy-momentum tensors is not a good idea and that the ordinary canonical superenergy tensors are better and more fundamental construction. The latter tensors are unambiguous and they do not “collide” with any energy-momentum tensor.

Recently we have observed the strong correlation between the sign of the total superenergy density  $\epsilon_s = \epsilon_g + \epsilon_m$  and stability of the solutions to the Einstein equations. Namely, we have noticed that the total superenergy density  $\epsilon_s$  is positive-definite or null for stable solution and negative-definite for unstable solutions. Thus we think that the following Conjecture is valid.

### Conjecture

Sign of the total superenergy density  $\epsilon_s$  determines stability or instability of a solution to the Einstein equations: if  $\epsilon_s \geq 0$ , then the solution is stable; when  $\epsilon_s < 0$ , then the solution is unstable.

We have not proved this Conjecture yet. Up to now we are only able to give examples which confirm it.[7]

### The examples

1. Exterior Schwarzschild:  $\epsilon_s > 0$  ——— stable;
2. Kerr solution:  $\epsilon_s > 0$  ——— stable;
3. Minkowski spacetime:  $\epsilon_s = 0$  ——— stable;
4. Friedman universes:  $\epsilon_s > 0$  ——— stable;
5. Kasner universe:  $\epsilon_s > 0$  ——— stable;
6. Bianchi I spacetime:  $\epsilon_s > 0$  ——— stable;
7. Anti-de Sitter spacetime:  $\epsilon_s < 0$  ——— unstable;
8. De Sitter spacetime:  $\epsilon_s < 0$  ——— unstable.

Instability of the de Sitter and anti-de Sitter spacetimes was proved recently [5].

## III. CONCLUSION

On the *superenergy level* or on the *averaged relative energy-momentum level* we have no problem with suitable tensor for gravity.

In our opinion, the canonical superenergy tensors seem more fundamental than the corresponding averaged relative energy-momentum tensors, e.g., they are independent of an dimensional factor  $L$ .

The canonical superenergy tensors are very useful to local analysis of the solutions to the Einstein equations; especially to analyse of their singularities.

Probably, these tensors give us also the very simple and powerful method to study stability of the solutions to the Einstein equations.

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  - [6] In terms of  $E_i{}^k$  Einstein equations read  $R_i{}^k = \beta E_i{}^k$ .
  - [7] We don’t know any counterexample.